Recent developments in Risk Theory

Inaugural Professorial Lecture

Prof. Franck Adékambi

School of Economics
College of Business and Economics

Johannesburg, 18 July 2023

Respondent: Prof. Runhuan Feng

Objective

- Describe my area of expertise, which is actuarial science.

Development of Actuarial Science

- According to Merriam-Webster (2001), an actuary calculates insurance premiums, reserves, and dividends. According to the Dictionnaire Robert historique de la langue française (1994), actuaire is a term derived from the latin actuarius, which was used in ancient Rome to refer to scribes (or secretaries, stenographers) who wrote the official proceedings.

- The actuary plays an important role in the "design", the management and the control of the insurance products, income, insurance schemes and collective insurance, as well as social security programs, with millions of people depending on their current and future financial security.

UNIVERSITY OF JOHANNESBURG

- Besides solid mathematical, statistical, and probability knowledge, an actuary should also be familiar with finance, economics, management, and marketing, regardless of any insurance or pension legislation.

- Villagers come to the aid of their fellow members when one of them is battered by an unforeseen event.

- It could be monetary or material aid, or it could even be the provision of bras approved to be used within the community to cultivate their fields for a short period of time.

- It was the primitive form of insurance that remains in effect in some African countries; however, with the advent of the industrial revolution, many members of these communities migrated into the cities to seek a better life for themselves.

- Once in the cities, this initial village solidarity collapsed because people rarely saw each other or spoke very little due to relatively primitive communications.

- During this period, the first insurance companies were formed, as workers obtained their lowest wages in the factories, which meant they had difficulty meeting their own needs and those of others in their community, as well as all the risks associated with industrial development.

- Actuaries first appeared in Great Britain at the end of the 19th century.

- Actuaries are in fact needed due to the industrial revolution, which created the need for large companies to define financial risks and put in place plans to protect themselves.

- Since the industrial revolution, there have been numerous forms of insurance designed to reduce risk.

UNIVERSITY OF JOHANNESBURG

- A growing number of actuaries are being asked to model these risks and develop the tools necessary to protect enterprises from them.

- In London, the first Institute of Actuaries formed around the turn of the nineteenth century, while in Edimbourg, a Faculty of Actuaries was established at the same time.

- Actuaries are primarily influenced by the maritime industry on this side of the Atlantic.

- The fact is that Nathaniel Bowditch was one of the pioneers in this field.

- Having been born in Salem in 1773, he worked as a clerk during his young years, which allowed him to travel across the ocean several times.

UNIVERSITY OF JOHANNESBURG

- At the age of 18, he had already copied all the mathematic writings he had come across concerning the Transactions of the Royal Society of London.

- In addition to his contributions to mathematical sciences, Nathaniel Bowditch also contributed to the development of physiology and astronomy theories.

- Later, he applied his science to a maritime insurance company as president, appointing him as the first American actuary.

- A few years later, or in 1889, the first association of the profession in North America was formed and called the "Actuarial Society of America".

- It was established in 1907 in Toronto that the first club of actuaries became the Association Canadienne des Actuaires, a non-constituted organization.

UNIVERSITY OF JOHANNESBURG

- The purpose of this association is to promote the knowledge of actuaries among its members.
- In 1965, the Association of Canadian Actuaries became the ICA, the current Canadian Institute of Actuaries.
- When the Association is created, all members are enrolled in the Institute.

While the role of the actuary has always primarily been to calculate insurance passifs, today's actuary can also deal with issues related to public policy.

Actuaries play a significant role in the establishment of our modern economy in Canada

UNIVERSITY OF JOHANNESBURG

Professional Status

- South Africa

- North America (USA & Canada)

- Actuaries in America are required to pass a series of exams in order to become licensed.. These exams are administered by two organizations: the Society of Actuaries (SOA) and the Casualty Actuarial Society (CAS). In essence, this system of professional examinations is intended to harm the knowledge levels and status levels of actuaries (associates and fellows). After passing the ensemble of exams, an actuary obtains the title of Fellow (of the SOA or CAS). Candidates or candidates who have completed part of the exams (e.g. two tiers) are referred to as Associate.

- SoA members work primarily in the fields of insurance and retirement income, health insurance, collective insurance, retirement and retirement plans, social security, investments, and finance.

- CAS members work in general insurance, which includes insurance for housing, automobiles, maritime, aviation, and liability. Additionally, he can work in social security and health insurance.

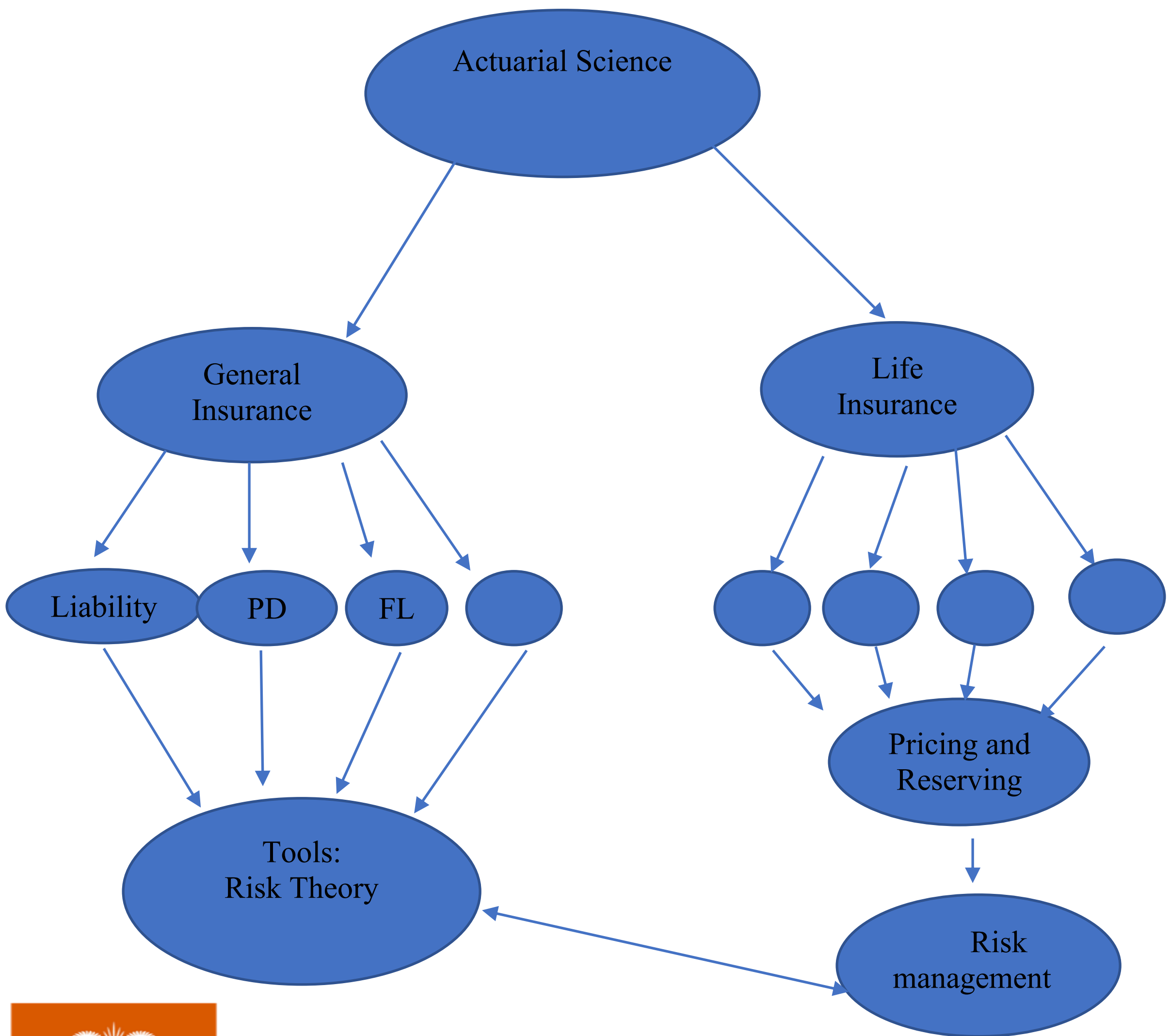
- UK

- There is a similar system of qualification in Grande-Bretagne and in Australia.

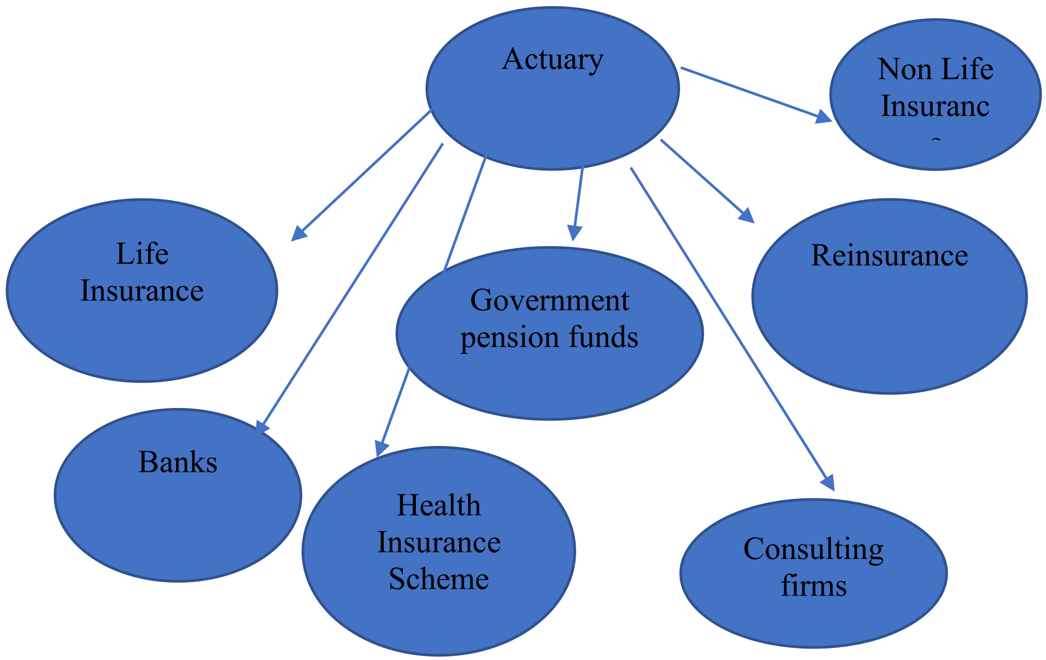
- In continental Europe, or in the absence of an actuarial science program, actuaries are declared qualified after completing a university program.

UNIVERSITY OF JOHANNESBURG

The different branches of actuarial science



- **The actuary's interventions**



My research interest

A- General insurance

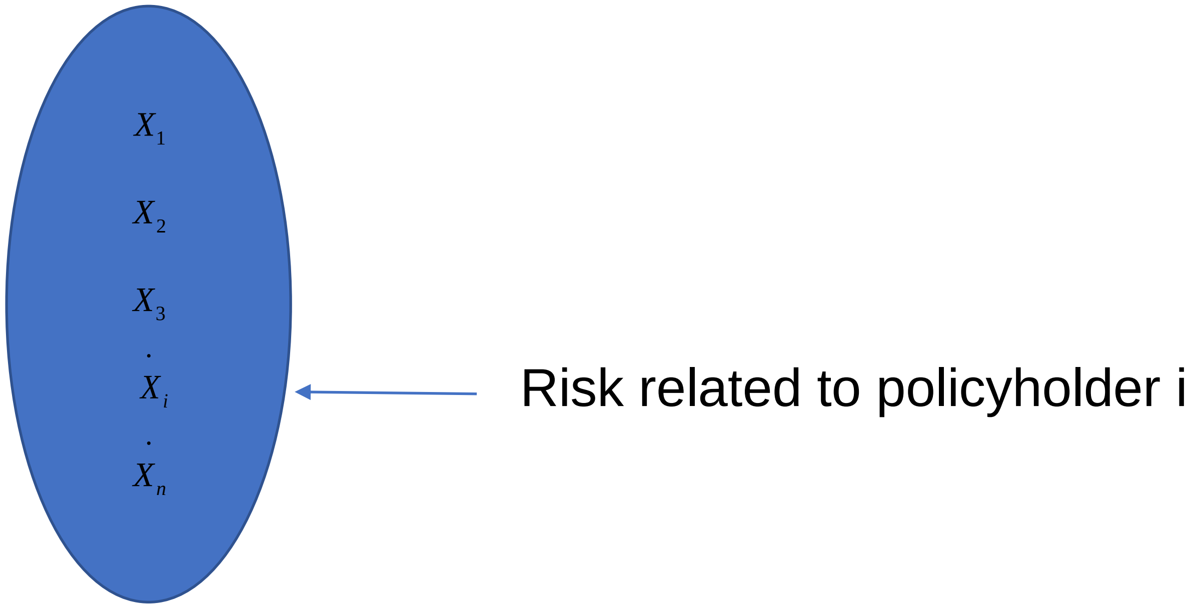
From the classic model to the discounted model

Collectif risk model

- Two models to represent the total amount of claims.
- In the "classic" individual risk model, claims are linked to each policy in a portfolio of size n.
- If $\{X_k , k \in \mathbb{N}\}$ is the claim amount associated with the kth policy, then the total claim amount for this portfolio is:

$$Z = \sum_{k=1}^{n} X_k$$

- where $\{X_k, k \in \mathrm{N}\}$ are non-negative and independent but not necessarily identically distributed random variables.



The total risk of the portfolio is given by

$$S^{ind} = X_1 + X_2 + \ldots + X_n = \sum_{i=1}^{n} X_i$$

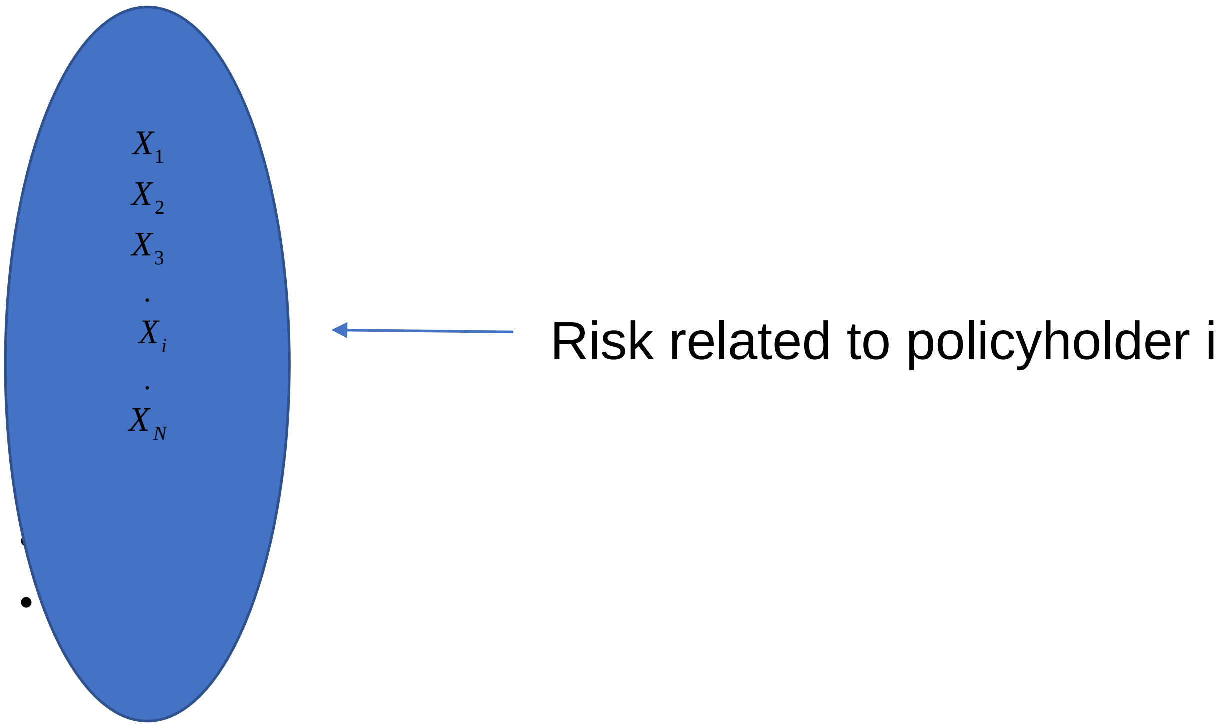
An actuary is interested in risk measures such as:

- Mean
- Variance, Value at Risk, more generally the moment generating function $(M_{S^{ind}}(r) = E\left[e^{rS^{ind}}\right])$

- This model is called individual risk model

Collective risk model

Another approach is called collective risk model



- In the collective risk model, risks are no longer considered individually but as several shocks that affect the portfolio.

- If $N(t)$ represents the number of claims in the period $[0, t]$, and $\{X_k, k \in \mathbb{N}\}$ is the amount of the kth claim, then we can write

$$Z = \sum_{k=1}^{N(t)} X_k \ ,$$

- $\{X_k, k \in \mathbb{N}\}$ positive random variables i.i.d, $N(t), X_1, X_2, \ldots$ v.a mutually independent, for any $t > 0$.

Importance of the economic environment

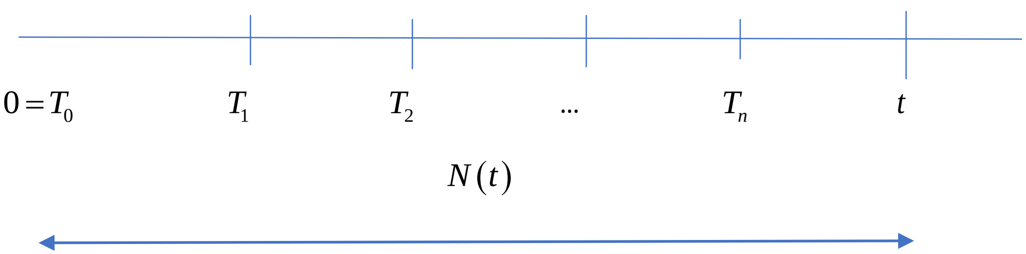
- The classic collective model, no account taken of inflation
- Inflation can have a significant impact on the amounts insured. Insurance companies need to know the effect of inflation on their total liability when calculating premiums or reserves.
- Inflation
  - Increase in the cost of claims
  - Increase in portfolio volume

UNIVERSITY OF JOHANNESBURG

An actuary is interested in risk measures such as:

- Mean
- Variance, Value at Risk, more generally the moment generating function $M_{S^{coll}}(r) = E\left[e^{rS^{coll}}\right]$
- The collective risk model doesn't consider time.

The first model, authors worked with was when is a Poisson process.



The model was later improved by considering that the time between each claim can follow any distribution.

Discounted renewal amounts with general interest rate

- Model Assumptions:

(i) The claims number process $\{N(t), t \geq 0\}$ forms an ordinary renewal process and, for $k \in \mathrm{N} = \{1,2,3,...\}$, i.e. :

- The time when the k-th claim occurs is represented by the positive variable $\{T_k, k \in \mathrm{N}\}$.

- the positive variable representing the time between two consecutive claims is given by $\tau_k = T_k - T_{k-1}$, with $T_0 = 0$ and common distribution function F_τ.

(ii) the variable representing the amount of the claim, without inflation, is given by $\{X_k , k \in \mathrm{N}\}$, where :

- $\{X_k , k \in \mathrm{N}\}$ are i.i.d,

UNIVERSITY OF JOHANNESBURG

- $\{X_k, \tau_k ; k \in \mathrm{N}\}$ are mutually independent.
- the moment generating function of X_1, M_X exist over $\Omega \subset \Re$,
- $0 < \mu_k = E[X_k] < \infty$

(iii) $\delta(t)$, deterministic or random force of interest.

(iv) The total discounted (net) value at time t of the total claims over the period $[0, t]$ is denoted respectively, for the ordinary and delayed case, by:

$$Z(t) = \sum_{k=1}^{N(t)} D(T_k) X_k \ ,$$

$$Z(t) = 0 \text{ si } N(t) = 0, \quad D(T_k) = \exp\left\{ -\int_0^{T_k} \delta(x) dx \right\},$$

- Higher Moments

Lemma 1.

Consider an ordinary or delayed renewal process, as described above. Then, for any $0 = x_0 < x_1 < x_2 < \ldots < x_k \le t$, $i_0 = 0$, $1 \le i_1 < i_2 < \ldots < i_k \le n$ and $1 \le k \le n$, the conditional joint distribution of $T_{i_1}, T_{i_2}, \ldots, T_{i_k} \mid N(t) = n$ is given by :

$$f_{T_{i_1}, T_{i_2}, \ldots, T_{i_k} \mid N(t)}(x_1, x_2, \ldots, x_k \mid n) = \frac{P\left(N\left(t - x_k\right) = n - i_k\right) \prod_{j=1}^{k} f_{T_{i_j - i_{j-1}}}\left(x_j - x_{j-1}\right)}{P(N(t) = n)} .$$

- Higher Moments

- Deterministic force of interest

Theorem 1

With the same assumptions as before, the first moment of the expected renewal sum is given, for $t>0$ and for a deterministic force of interest, by:

$$E\left[Z(t)\right]=E\left[X_1\right]\int_0^t D(v)\,dm(v)$$

Proof:

$$E\left[Z(t)\middle|N(t)=n\right]=E\left[X_1\right]\sum_{k=1}^{n}\int_0^t D(v)\,f_{T_k|N(t)=n}(v)\,dv$$

- Higher Moments

- **Stochastic force of interest**

Theorem 2

With the same assumptions as before, the first moment of the expected renewal sum is given, for $t > 0$ and for a stochastic force of interest, by:

$$E\left[Z(t)\right] = E\left[X_1\right]\int_0^t E\left[D(v)\right]dm(u)$$

Proof

$$E\left[Z(t)\middle|N(t)=n\right]=E\left[\sum_{k=1}^{n}D(T_k)X_k\middle|N(t)=n\right]$$

$$=E[X_1]E\left[\sum_{k=1}^{n}D(T_k)\middle|N(t)=n\right]$$

- For each sample path of $\delta(x)$:

$$E\left[Z(t)\middle|\delta(x),x\in[0,t]\right]=E\left[E\left[Z(t)\middle|N(t),\delta(x),x\in[0,t]\right]\right]$$

$$=E[X_1]\int_0^t D(v)\,dm(v)$$

- Last integral, a random variable, a well-known theorem of the theory of stochastic processes (see Karatzas (1991), P. 3)

Higher Moments

Example 1

- $\{\delta(t), t \geq 0\}$, an Itô process satisfying the stochastic differential equation of Ho-Lee-Merton

$$d\delta(t) = r\,dt + \sigma\, dB(t),$$

with constant drift r, volatility $\boldsymbol{\sigma}$, where $B(t)$ is a standard Brownian motion (see Oksendal (1992)).

- We have

$$E\left[D^2(u)\right] = \exp\left\{-\delta(0)v - \frac{rv^2}{2} + \sigma^2\frac{v^3}{6}\right\}$$

and,

$$E\left[Z(t)\right] = E\left[X_1\right]\int_0^t e^{-\delta(0)v - \frac{rv^2}{2} + \sigma^2\frac{v^3}{6}}\,dm(v).$$

When $\tau_k \sim \exp(\lambda = 1)$, $E[X_1] = 1$, $\delta(0) = 0.03$, $r = 0.002$ and $\sigma = 0.001$. Then, using the software "Maple", we have the table below

Table 1 : First moment of $Z(t)$ -- Cas Ho-Lee-Merton

t	1	5	10	15	20
$E[Z(t)]$	0.98482309 73	4.6061153 32	8.3806863 12	11.3241284 6	13.50862840
t	30	40	50	60	70
$E[Z(t)]$	16.089518	17.158952	17.524116	17.626955	17.650864

- Higher Moments

- Deterministic force of interest

Theorem 3

With the same assumptions as before, the first moment of the expected renewal sum is given, for $t > 0$ and for a deterministic force of interest, by:

$$E\left[Z^2(t)\right] = E\left[X_1^2\right]\int_0^t D^2(v)\,dm(v)$$

$$+E^2\left[X_1\right]\int_0^t\int_0^{t-v} D(u+v)D(v)\,dm(v)\,dm(u)$$

- Stochastic force of interest

Theorem

With the same assumptions as before, the first moment of the expected renewal sum is given, for $t > 0$ and for a stochastic force of interest, by:

$$E\left[Z^2(t)\right] = E\left[X_1^2\right]\int_0^t E\left[D^2(v)\right]dm(v)$$

$$+E^2\left[X_1\right]\int_0^t\int_0^{t-v} E\left[D(u+v)D(v)\right]dm(v)dm(u)$$

Example 2

- $\{\delta(t), t \geq 0\}$, an Itô process satisfying the stochastic differential equation of Ho-Lee-Merton

$$d\delta(t) = rdt + \sigma\, dB(t),$$

with constant drift r, volatility $\boldsymbol{\sigma}$, where $B(t)$ is a standard Brownian Motion (see Oksendal (1992)).

- We have

$$E\left[D^2(u)\right]=\exp\left\{-2\delta(0)v-rv^2+\frac{2}{3}\sigma^2v^3\right\},$$

$$E\left[D(u+v)D(v)\right]=\exp\left\{-\left[\delta(0)(u+2v)\right]-\frac{r}{2}\left[u^2+2uv+v^2\right]+\frac{\sigma^2}{2}\left[\frac{(u+2v)^3+u^3}{6}\right]\right\}$$

and,

$$E\left[Z(t)\right]=E\left[X_1^2\right]\int_0^t e^{-2\delta(0)v-rv^2+\frac{2}{3}\sigma^2v^3}dm(v)$$

$$+E^2\left[X_1\right]\int_0^t\int_0^{t-v} e^{-\left[\delta(0)(u+2v)\right]-\frac{r}{2}\left[u^2+2uv+v^2\right]+\frac{\sigma^2}{2}\left[\frac{(u+2v)^3+u^3}{6}\right]}dm(v)\,dm(u).$$

When $\tau_k\sim\exp(\lambda=1)$, $E\left[X_1\right]=1$, $\delta(0)=0.03$, $r=0.002$ and $\sigma=0.001$. Then, using the software "Maple", we have the table below

Table 2 Second moment of $Z(t)$ -- Cas Ho-Lee-Merton

t	1	5	10	15	20
$E\left[Z^2(t)\right]$	2.909785 123	29.72458 394	84.470666 79	145.9729 435	202.178562 1
t	30	40	50	60	70
$E\left[Z^2(t)\right]$	280.0771	315.9860	328.7405	332.3813	333.2318

- Joint moments

- Constant force of interest

Lemma 2

For the same risk process and for any $t > 0$, $h > 0$, $\delta \geq 0$ and $(u,v) \in \Omega \times \Omega$, the joint moments generating function between $Z(t)$ and $Z(t+h)$ satisfies the following equation

$$M_{Z(t),Z(t+h)}(u,v) = \bar{F}_{\tau_1}(t+h) + \int_t^{t+h} M_X\left(ve^{-\delta x}\right) M_{Z(t+h-x)}\left(ve^{-\delta x}\right) dF_{\tau_1}(x)$$

$$+ \int_0^t M_X\left((u+v)e^{-\delta x}\right) M_{Z(t-x),Z(t+h-x)}\left(ue^{-\delta x}, ve^{-\delta x}\right) dF_{\tau_1}(x) \ .$$

Theorem 5

For the same hypotheses, and for a constant force, the recurrence formula of the joint moment, for $n, m \in \mathrm{N}$, is given by

UNIVERSITY OF JOHANNESBURG

$$E\left[Z^n(t)\,Z^m(t+h)\right]=\sum_{k=1}^{n+m}E\left[X_1^k\right]\sum_{i=[k-m]_+}^{\min(k,n)}\binom{n}{i}\binom{m}{k-i}$$

$$\times\int_0^t e^{-(n+m)\delta u}E\left[Z^{n-i}(t-u)\,Z^{m-(k-i)}(t+h-u)\right]dm(u)\,.$$

- Stochastic force of interest

Theorem 6

For the same hypotheses, and for a stochastic force, the joint moment, of $Z(t)$ and $Z(t+h)$ are given, for $t>0$ and $h>0$, by

$$E\left[Z(t)Z(t+h)\right]=E\left[Z^2(t)\right]$$

$$+E^2\left[X_1\right]\int_0^t\int_{t-u}^{t+h-u}E\left[D(u)D(u+v)\right]dm(v)\,dm(u)\,,$$

- Moments generating function

- Deterministic force of interest

Theorem 7

UNIVERSITY OF JOHANNESBURG

$\forall\ t > 0, s \in \Omega$ and for a deterministic force of interest $\delta(t)$, the formula of the moments generating function is given by

$$M_{Z(t)}(s) = \bar{F}_{\tau_1}(t) + \sum_{n=0}^{\infty} \int_0^t \int_{u_1}^t \dots \int_{u_n}^t \prod_{i=1}^{n+1} M_X(sD(u_i)) \bar{F}_{\tau_1}(t - u_n) dF_{\tau_1}(u_{n+1} - u_n) \dots dF_{\tau_1}(u_1)$$

- Stochastic force of interest

Theorem

$\forall\ t > 0, s \in \Omega$ and for a stochastic force of interest $\delta(t)$, the formula of the moments generating function is given by

$$M_{Z(t)}(s) = \bar{F}_{\tau_1}(t) + \sum_{n=0}^{\infty} \int_0^t \int_{u_1}^t \dots \int_{u_n}^t E\left[\prod_{i=1}^{n+1} M_X(sD(u_i)) \right] \bar{F}_{\tau_1}(t - u_n) dF_{\tau_1}(u_{n+1} - u_n) \dots dF_{\tau_1}(u_1)$$

Two related applications

- Predictors

Linear predictor

Our linear predictor can therefore be rewritten in the following form:

$$L(t,h)=E\left[Z(t+h)\right]+\rho(t,h)\left[\frac{V\left[Z(t+h)\right]}{V\left[Z(t)\right]}\right]^{1/2}\left[Z(t)-E\left[Z(t)\right]\right].$$

Now consider the special case where the amount of claims follows a degenerate distribution at 1, the number of claims follows a Poisson distribution with parameter $\lambda=1$ and $\delta=0.005$. Then, taking into account the identities already obtained, either

$$E[Z(t)]=\frac{1-e^{-\delta t}}{\delta}\ ,\ Var[Z(t)]=\frac{1-e^{-2\delta t}}{2\delta}\ ,\ \rho(t,h)=\left[\frac{1-e^{-2\delta t}}{1-e^{-2\delta(t+h)}}\right]^{1/2},$$

we can first simulate the value of $Z(t)$, then we compare the simulated value of $Z(t+h)$ with that of $L(t,h)$ in the following table, for different values of t and of h.

Table 3 Comparison between $Z_{simul}(t+h)\big|Z(t)$ and $L(t,h)$

t	h	$Z(t)$	$Z_{simul}(t+h)\big|Z(t)$	$L(t,h)$
1	0.01	0.967	1.007	1.012885876
1	1	0.967	1.927	1.995465089
1	10	0.967	11.011	10.70840225
10	0.01	9.985	9.999	9.994512056
10	1	9.985	11.113	10.93385531
10	10	9.985	20.037	19.26340129
100	0.01	100.1	100.241	100.1060652
100	1	100.1	101.136	100.7050169
100	10	100.1	110.077	106.016169

Quadratic predictor

$$Q(t,h) = a + bZ(t) + cZ^2(t)$$

$$\begin{aligned}\Delta_a = {} & E\left[Z(t+h)\right]\left\{E\left[Z^2(t)\right]E\left[Z^4(t)\right] - E^2\left[Z^3(t)\right]\right\} \\ & - E\left[Z(t)Z(t+h)\right]\left\{E\left[Z(t)\right]E\left[Z^4(t)\right] - E\left[Z^2(t)\right]E\left[Z^3(t)\right]\right\} \\ & + E\left[Z(t)Z(t+h)\right]\left\{E\left[Z(t)\right]E\left[Z^4(t)\right] - E\left[Z^2(t)\right]E\left[Z^3(t)\right]\right\}\end{aligned}$$

$$\begin{aligned}\Delta_b = {} & E\left[Z(t)Z(t+h)\right]E\left[Z^4(t)\right] - E\left[Z^2(t)Z(t+h)\right]E\left[Z^3(t)\right] \\ & - E\left[Z(t)\right]\left\{E\left[Z(t+h)\right]E\left[Z^4(t)\right] - E\left[Z^2(t)Z(t+h)\right]E\left[Z^2(t)\right]\right\} \\ & + E\left[Z^2(t)\right]\left\{E\left[Z(t+h)\right]E\left[Z^3(t)\right] - E\left[Z^2(t)\right]E\left[Z(t)Z(t+h)\right]\right\}\end{aligned}$$

$$\Delta_c = E\left[Z^2(t)Z(t+h)\right]E\left[Z^2(t)\right] - E\left[Z(t)Z(t+h)\right]E\left[Z^3(t)\right]$$
$$- E\left[Z(t)\right]\left\{E\left[Z^2(t)Z(t+h)\right]E\left[Z(t)\right] - E\left[Z(t+h)\right]E\left[Z^3(t)\right]\right\}$$
$$+ E\left[Z^2(t)\right]\left\{E\left[Z(t+h)Z(t)\right]E\left[Z(t)\right] - E\left[Z^2(t)\right]E\left[Z(t+h)\right]\right\}$$

and,

$$a = \frac{\Delta_a}{\Delta}\ ,\ b = \frac{\Delta_b}{\Delta}\ ,\ c = \frac{\Delta_c}{\Delta} \quad .$$

Table 4 Comparison between $Z_{simul}(t+h)|Z(t)$ and $Q(t,h)$

t	t	$Z(t)$	$Z_{simul}(t+h)\|Z(t)$	$Q(t,h)$
1	0.001	0.967	1.007	1.010867476
1	1	0.967	1.927	1.945560054
1	10	0.967	11.011	10.80720814
10	0.01	9.985	9.999	9.997016090
10	1	9.985	11.113	11.09221136
10	10	9.985	20.037	19.54650814
100	0.01	100.1	100.241	100.1844870
100	1	100.1	101.136	100.9025264
100	10	100.1	110.077	108.223459

-

- Predictor
  Lundberg Inequality-Cai-Dickson

To the Sparre Andersen model defined in Chapter 2, Cai & Dickson incorporate the constant interest force. They consider that the insurer receives interest on its surplus process at a constant force $\delta > 0$. This surplus process noted $U_\delta(t)$ is given by

$$U_\delta(T_1) = ue^{\delta\tau_1} + \pi\left(\frac{e^{\delta\tau_1} - 1}{\delta}\right) - Y_1$$

.

.

.

$$U_\delta(T_n) = U_\delta(T_{n-1})e^{\delta\tau_n} + \pi\left(\frac{e^{\delta\tau_n} - 1}{\delta}\right) - Y_n$$

$$= ue^{\delta T} + \pi\left(\frac{e^{\delta T_n} - 1}{\delta}\right) - \sum_{k=1}^{n} X_k \exp\left\{-\delta\sum_{i=1}^{k}\tau_i\right\}$$

$$U_\delta(T_1) = ue^{\delta\tau_1} + \pi\left(\frac{e^{\delta\tau_1} - 1}{\delta}\right) - Y_1$$

.

.

.

$$U_\delta(T_n) = U_\delta(T_{n-1})e^{\delta\tau_n} + \pi\left(\frac{e^{\delta\tau_n} - 1}{\delta}\right) - Y_n$$

$$= ue^{\delta T} + \pi\left(\frac{e^{\delta T_n} - 1}{\delta}\right) - \sum_{k=1}^{n} X_k \exp\left\{-\delta\sum_{i=1}^{k}\tau_i\right\}$$

Suppose that the failure time of this modified surplus process is $\tau_\delta = \inf\left\{t : \left(U_\delta(t) < 0\right)\right\}$, where $U_\delta(t)$ is the surplus at time t and $U_\delta(0) = u$ We note $\psi_\delta(u)$ the ultimate probability of time to ruin when the force of interest is δ. Then,

$$\psi_\delta(u) = P\{\tau_\delta < \infty\}$$

Since ruin can only occur upon a claim, we have :

$$\psi_\delta(u) = P\left\{\bigcup_{n=1}^{\infty}\left(U_\delta(T_n) < 0\right)\right\}$$

For a constant force of interest: $\delta > 0$, , we note:

$$\bar{s}_{\overline{t}|\delta} = \int_0^t e^{-\delta\alpha} d\alpha = \frac{1 - e^{-\delta t}}{\delta}$$

.

Further on, if we define the quantities :

$$\psi_\delta(u;n) = P\left\{\bigcup_{k=1}^{n}\left(U_\delta(T_k) < 0\right)\right\}$$

then

$$\lim_{n\to\infty} \psi_\delta(u;n) = \psi_\delta(u)$$

UNIVERSITY OF JOHANNESBURG

B- Multistate health care model

- A quick backgroud.

- An example of multistate health care model is when an individual is healthy then becomes invalid and he may recover from his invalidity or died.

- What we mean by durational effect is how long the individual stay in each state.

- Norberg in 1995 used the same multistate health care model but without durational effect to derive the distribution of the discounted

B-1 Multistate model for the health status

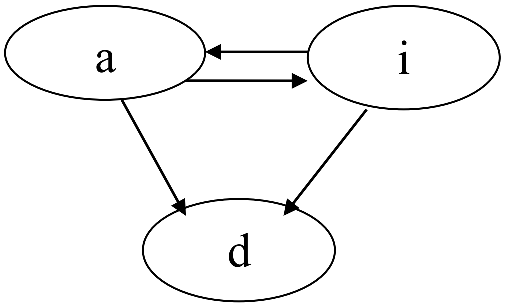
The random pattern of states of the policyholder

For health insurance, contractual guaranteed payments between insurer and policyholder are defined as deterministic function of time and of the pattern of states of the policyholder. Before we introduce a general modeling framework for that pattern of states, we give two examples of customary health insurance contracts.

Example1 (disability insurance)

A disability insurance or permanent health insurance (PHI) provides an insured with an income if the insured is prevented from working by disability due to sickness or injury. It is usually modeled by a multiple state model with state space

$$S := \{a = active / healthy, i = invalid / disabled, d = dead\}$$

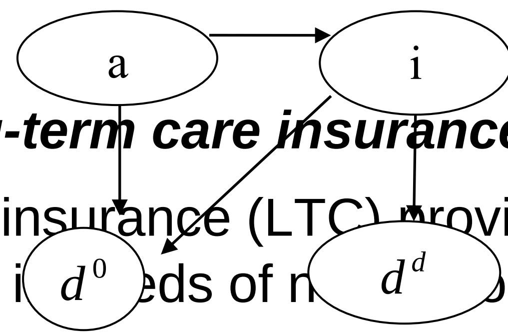


Disability insurance may be categorized as life or pension insurance rather than health insurance.

Example2 (critical illness insurance)

A critical illness insurance or dread disease insurance (DD) provides the policyholder with a lump sum if the insured contracts illness included in a set of diseases specified by the policy conditions. The most common diseases are heart attack, coronary artery disease requiring surgery, cancer and stroke. For example, it can be modeled by a multistate structure with space

$$S := \left\{ \begin{array}{l} a = active \,/\, healthy,\; i = invalid \,/\, disabled, \\ d^{d} = dead\;\; due\; to\; dread\;\; disease, \\ d^{0} = dead\;\; due\; to\; other\;\; causes \end{array} \right\}$$



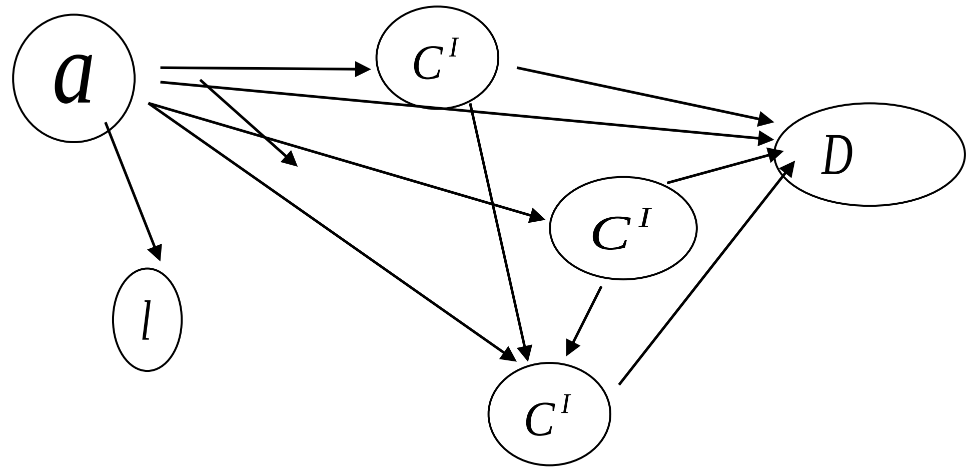
Example3 (Long-term care insurance)

A long-term care insurance (LTC) provides financial support for insureds who are i[illegible]eds of n[illegible]r medical care.

The needs for care due to the frailty of an insured is classified according to the individual’s ability to take care of himself by performing activities of daily living such as eating, bathing, moving around, going to toilet, or dressing.

LTC policies are commonly modeled by multistate models, and the state space usually consists of the states actives, dead, and the corresponding levels of frailty. For example in Germany three different levels of frailty are used and, moreover, lapse plays an important role. Thus, we have a state space of

$$S := \left\{ \begin{array}{l} a = active/healthy,\ C^{I} = need\ \ for\, basic\ care, \\ C^{II} = need\ \ for\, medium\ care, \\ C^{I} = need\ \ for\, comprehensive\ care, \\ l = lapse/cancel,\ \ d = dead \end{array} \right\}$$



Let the random pattern of states of an individual policyholder be given by a pure jump process $\left(\Omega, \Im, P, (X_t)_{t\geq 0}\right)$ with finite state space S and right continuous paths with left-hand limits, representing the state of the policy at time $t \geq 0$. We further define the transition space $J := \{(i,\ j) \in S \times S \mid i \neq j\}$, the counting processes

$$N_{jk}(t) := \#\{\boldsymbol{\tau} \in (0,t] \mid X_{\tau} = k, X_{\tau-} = j\}, (j,k) \in J$$

the time of the next jump after t

$$T(t) := \min\{\boldsymbol{\tau} > t \mid X_{\tau} \neq X_{\tau-}\},$$

the series of the jump times

$$S_0 := 0,\ S_n := T(S_{n-1}),\ n \in \mathbb{N},$$

and a process that gives for each time the time elapsed since entering the current state,

$$U_t := \max\{\boldsymbol{\tau} \in [0, t] \mid X_u = X_t \ \ for\ all\ u \in [t - \boldsymbol{\tau}, t]\},$$

also denoted as duration process. Instead of using a jump process $(X_t)_{t\geq 0}$, some authors describe the random pattern of states by a chain of jumps. The two concepts are equivalent.

- The semi-Markovian approach

The random pattern of states $(X_t)_{t\geq 0}$ is called semi-Markovian, if the bivariate process $(X_t, U_t)_{t\geq 0}$ is a Markovian process, which means that for all $i \in S$, $u \geq 0$, and $t \geq t_n \geq \ldots t_1 \geq 0$ we have

$$P\Big(\big(X_t, U_t\big) = \big(i, u\big)\Big| X_{t_n}, U_{t_n}, \ldots, X_{t_1}, U_{t_1}\Big) = P\Big(\big(X_t, U_t\big) = \big(i, u\big)\Big| X_{t_n}, U_{t_n}\Big)$$

almost surely. In the following we always assume that the initial state (X_0, U_0) is deterministic. (Note that $U_0 = 0$ by definition). In practice that means that we know the state of the policyholder when signing the contract. With this assumption and the Markov property for $(X_t, U_t)_{t \geq 0}$ we have that the probability distribution of $(X_t, U_t)_{t \geq 0}$ is already uniquely defined by the transitions probability matrix

$$p(s,t,u,v) = \big(P\big(X_t = k, U_t \leq v \big| X_s = j, U_s = u\big)\big)_{(j,k) \in S^2},$$

$$0 \leq u \leq s \leq t < \infty, v \geq 0.$$

Alternatively, we can also uniquely define the probability distribution of $(X_t, U_t)_{t \geq 0}$ by specifying the probabilities

$$\bar{p}(s,t,u) = \left(\bar{p}_{jk}(s,t,u)\right)_{(j,k)\in S^2},$$

$$\bar{p}_{jk}(s,t,u) := P\left(T(s) \le t,\ X_{T(s)} = k \,\middle|\, X_s = j, U_s = u\right),\ \ j \ne k,$$

$$\bar{p}_{jj}(s,t,u) := -P(T(s) \le t \,|\, X_s = j, U_s = u).$$

A third way to uniquely define the probability distribution of $(X_t, U_t)_{t\ge 0}$ is to specify the cumulative transition intensity matrix

$$q(s,t) = \left(q_{jk}(s,t)\right)_{(j,k)\in S^2},$$

$$q_{jk}(s,t) := \int_{(s,t]} \frac{\bar{p}_{jk}(s, d\boldsymbol{\tau}, 0)}{1 - \bar{p}_{jj}(s, \boldsymbol{\tau}-, 0)},\ \ 0 \le s \le t < \infty.$$

If $q(s,t)$ is differentiable with respect to t, we can also define the transition intensity matrix

UNIVERSITY OF JOHANNESBURG

$$\mu_{jk}(t, t-s) := \frac{d}{dt} q(s,t) = \left(\frac{\frac{d}{dt}\overline{p}_{jk}(s,t,0)}{1-\overline{p}_{jj}(s,t,0)} \right)_{(j,k)\in S\times S},$$

- which is some form of multistate hazard rate. The quantity $\mu_{jk}(t, t-s)$ gives the rate of transitions from state j to state k at time given that the current duration of stay in j is $t-s$.

B-2 The health insurance contract

Payments between insurer and policyholder are two types:

(a) The amount $b_{jk}(t,u)$ is payable if the policy jumps from state j to state k at time t and the duration of stay in j state was u. In the Markovian approach the parameter u plays no role, and we write $b_{jk}(t,u) = b_{jk}(t)$.

UNIVERSITY OF JOHANNESBURG

In order to distinguish between payments from insurer to insured and vice versa, benefit payments get a positive sign and premium payments get a negative sign.

(b) Annuity payments fall due during sojourns in a state and are defined by deterministic functions $B_j(s,t),\ j \in S$. Given that the last transition occurred at time s, $B_j(s,t)$ is the total amount paid in $[s,t]$ during a sojourn in state j. We assume that the $B_j(s,.)$ are right continuous and of bounded variation on compacts.

We assume that all contractual payments happen only on the time interval $[0,n]$. In insurance practice, n is for example the maximum age of a life table.

By statute the insurer must at any time maintain a reserve in order to meet future liabilities in respect of the contract. This reserve bears interest with some rate $\varphi(t)$. On the basis of this interest rate we define a discounting function,

UNIVERSITY OF JOHANNESBURG

$$v(s,t) := e^{-\int_s^t \varphi(r)dr}.$$

We can interpret $v(s,t)$ as the value at time s of a unit payable at time $t \geq s$. Next, we study the present value of future payments between insurer and policyholder, that is, the discounted sum of all future benefit and premiums payments,

$$A(t) := \sum_{j\in S}\sum_{l=0}^{\infty}\int_{(t,n]} v(t,\tau)1_{\{S_l \leq \tau < S_{l+1}\}} B_j(S_l, d\tau)$$
$$+ \sum_{(j,k)\in J}\int_{(t,n]} v(t,\tau) b_{jk}(\tau, U_\tau) dN_{jk}(\tau).$$

The quantity $A(t)$ is the amount that an insurer needs at time t in order to meet all future obligations in respect of the contract. Since we assumed that there are no payments after time n, we have $A(t) = 0$ for $t > n$.

Linking our paper to Norberg (1995), we may alternatively write

$$A(t) = \int_{(t,n]} v(t,\boldsymbol{\tau})\, dB(\boldsymbol{\tau}),$$

where from Norberg's definition, $B(t)$ is the random total amount paid in the time interval $[0, t]$.

Conclusions

- Extension of the results obtained by Léveillé and Garrido regarding expected renewal amounts.
- We have proposed explicit formulas for the first two moments of the discounted renewal sum with a general strength of interest as well as the joint moment of our risk model.
- Developed explicit formulas for higher order joint moments when the force of interest is more general and recurrence formulas for the joint moments of our risk process.
- An expression of the moment generating function of the renewal sum was found for deterministic forces of interest.
- An upper bound for the probability of ruin

UNIVERSITY OF JOHANNESBURG