An Upper Bound on the Radius of a 3-Edge-Connected Graph

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Abstract

Let $G$ be a 3-edge-connected graph of order $n$ and radius $\text{rad}(G)$. Then the inequality

$$\text{rad}(G) \leq \frac{1}{3}n + \frac{17}{3}$$

is proved. Moreover, graphs are constructed to show that the bound is asymptotically sharp.

1 Introduction

Let $G = (V(G), E(G))$ be a finite, simple, connected undirected graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices $u, v$ of $G$, $d_G(u, v)$, is the length of a shortest $u - v$ path in $G$. The eccentricity, $\text{ex}(v)$, of a vertex $v \in V$ is the maximum distance between $v$ and any other vertex in $G$. Every vertex of $G$ of minimum eccentricity is a centre vertex of $G$ and the eccentricity of a centre vertex is called the radius of $G$, denoted by $\text{rad}(G)$. The radius of a graph (a convenient model of a network) is an important measure of centrality. The central vertices in a network are of particular interest because they can play the role of organizational hubs. For instance, if one wishes to locate an emergency response facility, like a fire station, or a hospital, then the primary concern may be to choose a location such that the travel time/distance from the emergency facility

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to a location farthest away is as small as possible. The radius is a good measure that indicates the travel time from an emergency facility to a location farthest away, if the best location for the emergency facility is chosen. In metabolic networks, eccentricity and hence radius solve the tasks of efficiently allocating resources to required metabolites and of controlling a cell’s biochemistry with as little time delay as possible. In [16], applications of eccentricity as a measure of centrality to three different types of biological networks, namely metabolic, protein and domain sequence networks are discussed. For results in facility location problems and social networks, see for example [15].

The degree, \( \deg(v) \), of a vertex \( v \) of \( G \) is the number of edges incident with it. The minimum degree, \( \delta(G) \), of \( G \) is the minimum of the degrees of vertices in \( G \). The open neighbourhood of a vertex \( v \), \( N(v) \), is the set of all vertices of \( G \) adjacent to \( v \). The closed neighbourhood of \( v \), \( N[v] \), is the set \( N(v) \cup \{v\} \). The vertex-connectivity (edge-connectivity), \( \kappa(G) (\lambda(G)) \), of \( G \) is defined as the minimum number of vertices (edges) whose deletion from \( G \) results in a disconnected or trivial graph. \( G \) is \( k \)-connected (\( k \)-edge-connected) if \( \kappa(G) (\lambda(G)) \geq k \). For notions not defined here we refer the reader to [1].

Several upper bounds on radius in terms of other graph parameters are known. For example, the computer programme Graffiti [10] conjectured that the radius of a graph is not more than its independence number. This was successfully proved by various authors (e.g. [11, 12]) and a slightly stronger result can be found in [8]. In [9], radius-critical graphs are characterized. Erdős, Pach, Pollack, and Tuza [7] proved that if \( G \) is a connected graph of order \( n \) and minimum degree \( \delta \geq 2 \), then

\[
\text{rad}(G) \leq \frac{3(n - 3)}{2(\delta + 1)} + 5,
\]

and also constructed graphs that, apart from the additive constant, attain the bound. Moreover, they gave improved bounds for \( C_3 \)-free and \( C_4 \)-free graphs. Using different methods, Dankelmann, Dlamini and Swart [2, 3, 6] obtained the slightly stronger bound

\[
\text{rad}(G) \leq \frac{3n}{2(\delta + 1)} + 1. \tag{1}
\]

Bounds on the radius in terms of order and vertex-connectivity were given by Harant and Walther [14]. If \( \kappa(G) \) is even, then the well known bound \( \text{diam}(G) \leq (n + \kappa(G) - 2)/\kappa(G) \) on the diameter, defined as the maximum of all distances between two vertices of \( G \), is also sharp for the radius. For
odd $\kappa(G)$, Harant and Walther [14] showed that
\[
\text{rad}(G) \leq \frac{n}{\kappa(G)} + 1 + O(\log n).
\]
In [13], Harant showed that for $\kappa = 3$, the $O(\log n)$ term can be replaced by a constant. In this paper we give asymptotically sharp upper bounds on the radius of a graph of given order and edge-connectivity, $\lambda \geq 2$. For $\lambda \neq 3$, these bounds are a direct consequence of (1), while the case $\lambda = 3$ is more complicated.

**Definition 1** For positive integers $\pi_0, \pi_1, \ldots, \pi_{t-1}, k$ with $3 \leq k$ and $t \leq k$, $C_k(\pi_0, \pi_1, \ldots, \pi_{t-1})$ is the graph obtained from the $k$-cycle $C_k = v_0v_1v_2\cdots v_{k-1}v_0$ by replacing every vertex $v_i$ by the complete graph $K_{\pi_i}$ if $i \equiv j \mod t$ and $j \in \{0, 1, \ldots, t-1\}$, and making every vertex in $K_{\pi_r}$ adjacent to every vertex in $K_{\pi_s}$ whenever $K_{\pi_r}$ and $K_{\pi_s}$ have replaced adjacent vertices of $C_k$.

![Figure 1: $C_{12}(2, 2, 1)$](image)

**Proposition 1** Let $G$ be a $\lambda$-edge-connected graph, $\lambda \geq 2$, of order $n$.
(a) Then the radius of $G$ satisfies $\text{rad}(G) \leq \frac{3}{2} \left(\frac{n}{\lambda+1}\right) + 1$.
(b) Apart from an additive constant, the bound in (a) is best possible for $\lambda \geq 2$ and $\lambda \neq 3$. 
Lemma 1 \[ \text{Let} \]

The following useful observation is due to Erdős that

\[ d \]

Definition 2 \[ \text{Throughout this paper,} \]

\[ u, v \]

\[ z \]

\[ y \]

\[ z \]

\[ u, v \]

\[ \text{vertices} \]

\[ z \]

\[ x \]

\[ T \]

\[ G \]

\[ x \]

\[ g \]

\[ N \]

\[ G \]

\[ 2 \]

\[ \text{asymptotically sharp upper bound on the radius of a 3-edge-connected graph of given order.} \]

2 Results

Let \( G \) be a 3-edge-connected graph of order \( n \). Let \( z \) be a fixed centre vertex of \( G \) so that \( rad(G) = r = ex(z) \). For each \( i = 0, 1, \ldots, r \), let \( N_i := \{ v \in V(G) \mid d_G(v, z) = i \} \) and \( k_i = |N_i| \). We employ the notation \( N_{i,j} = \cup_{0 \leq i \leq j} N_i \) and \( N_{j} = \cup_{i \leq j} N_i \). Since \( N_r \neq \emptyset \), from now onwards fix a vertex \( z_r \in N_r \). Form a spanning tree \( T \) of \( G \) that is distance preserving from \( z \). For a vertex \( y \in V(G) \), denote by \( T(z, y) \), the set of vertices on a path connecting \( z \) and \( y \) in \( T \).

Definition 2 Let \( y \in V(G) \). We say \( y \) is related to \( z \), if there exist vertices \( u, v \in V(G) \), where \( u \in T(z, z_r) \cap N_{\geq 9} \) and \( v \in T(z, y) \cap N_{\geq 9} \) such that \( d_G(u, v) \leq 4 \).

The following useful observation is due to Erdős et al. [7].

Lemma 1 [7] Let \( rad(G) \geq 10 \) and \( z, z_r \) as above. Then there exists a vertex in \( N_{\geq r-9} \) which is not related to \( z_r \).

Throughout this paper, \( y_0 \) is a fixed vertex in \( N_{\geq r-9} \) which is not related to \( z_r \). Let \( T(z, z_r) = \{ z, x_1, x_2, \ldots, x_r \} \) \( (T(z, y_0) = \{ z, y_1, y_2, \ldots, y_r \} \) be a path from \( z \) to \( z_r \) \( (y_0) \) in \( T \). For each \( i \) with \( 0 \leq i \leq r \) \( (0 \leq i \leq r-9) \), denote by \( N_i' \) \( (N_{i}'' \) the set of all elements in \( N_i \) whose distance from at least one vertex of \( T(z, z_r) \cap N_{\geq 9} \) \( (T(z, y_0) \cap N_{\geq 9}) \) is at most 2 in \( G \). Let \( X_i \subseteq N_i' \) \( (Y_i \subseteq N_i'') \) denote the set of vertices whose distance from at least one vertex on \( T(z, z_r) \cap N_{\geq 9} \) \( (T(z, y_0) \cap N_{\geq 9}) \) is at most 1 in \( G \). Note that for all \( i \in \{ 7, \ldots, r-9 \} \), \( |N_i'|, |N_i''| \geq 1 \). The following facts follow from
the fact that $y_0$ is not related to $z_r$:

**Fact 1:** $(\bigcup_{i=1}^{7} N_i') \cap (\bigcup_{i=7}^{10} N_i'') = \emptyset \text{ and } k_i \geq 2 \text{ for all } i \in \{7, \ldots, r-9\}.$

**Fact 2:** There are no edges between $N_i'$ ($N_i''$) and $Y_j$ ($X_j$) for all $i, j \in \{7, \ldots, r-9\}.$

**Lemma 2** Let $\lambda(G) \geq 3$, $rad(G) \geq 10$. With the above notation:

(a) For each $i \in \{7, \ldots, r-10\}$, $k_i + k_{i+1} \geq 5$.

(b) If for some $i \in \{8, \ldots, r-10\}$, $k_i = 2$, then either $k_{i+1} \geq 4$ or $k_{i-1} \geq 4$.

(c) If for some $i \in \{8, \ldots, r-10\}$, $k_{i-1} = 2 = k_{i+1}$, then $k_i \geq 4$.

**Proof:** (a) Since each $k_i \geq 2$, it is sufficient to show that it is impossible to have $k_i = k_{i+1} = 2$. So suppose that $N_i = \{x_i, y_i\}$ and $N_{i+1} = \{x_{i+1}, y_{i+1}\}$.

By Fact 2, $x_i y_{i+1}, y_i x_{i+1} \notin E(G)$ hence $G$ can be disconnected by removing the edges $y_i y_{i+1}$ and $x_i x_{i+1}$, contradicting the fact that $G$ is 3-edge-connected.

(b) Assume that $k_i = 2$. Then by part (a) above, $k_{i-1} \geq 3$ and $k_{i+1} \geq 3$.

We show that the situation $k_{i-1} = 3, k_i = 2, k_{i+1} = 3$ is impossible. Suppose to the contrary that $N_{i-1} = \{u, x_{i-1}, y_{i-1}\}$, $N_i = \{x_i, y_i\}$ and $N_{i+1} = \{v, x_{i+1}, y_{i+1}\}$. Since $\deg(x_i), \deg(y_i) \geq 3$, we can assume that, without loss of generality, $N(x_i) = \{x_{i-1}, x_{i+1}, u\}, N(y_i) = \{y_{i-1}, y_{i+1}, v\}$.

Since $y_0$ is not related to $z_r$, $N[x_i]$ and $N[y_i]$ are disjoint and not joined by an edge. Hence the removal of $y_i y_{i+1}$ and $x_i x_{i+1}$ disconnects $G$, which is a contradiction.

(c) Assume that $k_{i-1} = 2 = k_{i+1}$. Let $\{x_i, y_i\} \subseteq N_i$. Since $y_0$ is not related to $z_r$, the closed neighbourhoods $N[x_i], N[y_i]$ are disjoint. Hence $2(\delta + 1) \leq |N[x_i]| + |N[y_i]| \leq k_{i-1} + k_i + k_{i+1}$. In conjunction with $\delta \geq 3$ and $k_{i-1} = k_{i+1} = 2$, we obtain $k_i \geq 4$.

**Definition 3** If $l \in \{8, \ldots, r-11\}$ and $k_l + k_{l+1} = 5$, we call $(N_l, N_{l+1})$, a 5-class.

**Lemma 3** Let $\lambda(G) \geq 3$, $rad(G) \geq 10$ and let $(N_l, N_{l+1})$ $(8 \leq l \leq r-11)$ be a 5-class. Then the inequality

$$k_{l-1} + k_l + k_{l+1} + k_{l+2} \geq 12$$

holds. Moreover, if $e$ is the least positive integer such that $(N_{l+e}, N_{l+e+1})$ is a 5-class, then $e \geq 4$.

**Proof:** If $k_2 = 2$ and $k_{l+1} = 3$, then by Lemma 2 (b) $k_{l-1} \geq 4$. By Lemma 2 (c), $k_{l+2} \geq 3$. Hence $k_{l-1} + k_l + k_{l+1} + k_{l+2} \geq 4 + 4 + 3 = 12$. On the other hand if $k_3 = 3$ and $k_{l+1} = 2$, then by Lemma 2 (b), $k_{l+2} \geq 4$. By Lemma 2 (c), $k_{l-1} \geq 3$. Hence $k_{l-1} + k_l + k_{l+1} + k_{l+2} \geq 3 + 3 + 2 + 4 = 12$ as desired. It remains to show the second part of the lemma. We prove this result in three parts:
PART A: If \((N_l, N_{l+1})\) is a 5-class, then \((N_{l+1}, N_{l+2})\) is not a 5-class

If \(k_l = 2\) and \(k_{l+1} = 3\), then by Lemma 2 (c), we must have \(k_{l+2} \geq 3\), so that \((N_{l+1}, N_{l+2})\) is not a 5-class. On the other hand, if \(k_l = 3\) and \(k_{l+1} = 2\), then by Lemma 2 (b), we must have \(k_{l+2} \geq 4\), hence \((N_{l+1}, N_{l+2})\) cannot be a 5-class.

PART B: If \((N_l, N_{l+1})\) is a 5-class, then \((N_{l+2}, N_{l+3})\) is not a 5-class

Case 1: \(k_l = 3\), \(k_{l+1} = 2\). By Lemma 2 (b), we must have \(k_{l+2} \geq 4\). Therefore, by \(k_{l+3} \geq 2\), \((N_{l+2}, N_{l+3})\) cannot be a 5-class.

Case 2: \(k_l = 2\), \(k_{l+1} = 3\). By Lemma 2 (b), we must have \(k_{l+2} \geq 3\). Therefore, \((N_{l+2}, N_{l+3})\) can only be a 5-class if \(k_{l+2} = 3\) and \(k_{l+3} = 2\). We show that this situation, i.e., \(k_l = 2\), \(k_{l+1} = 3\), \(k_{l+2} = 3\), \(k_{l+3} = 2\) is forbidden. Suppose \(N_l = \{x_l, y_l\}, N_{l+1} = \{x_{l+1}, y_{l+1}, u\}, N_{l+2} = \{x_{l+2}, y_{l+2}, v\}\) and \(N_{l+3} = \{x_{l+3}, y_{l+3}\}\). Since \(k_l = 2\), \(u\) is adjacent to either \(y_l\) or to \(x_l\) (but not both). Thus either \(u \in X_l\) or \(u \in Y_l\). Assume that \(u \in X_l\) (The case \(u \in Y_l\) is proved analogously). With this condition and that \(k_{l+1} = 3\) it follows that either \(v \in N'_{l+2}\) or \(v \in N''_{l+2}\). We have two subcases to look at:

Subcase 1: \(u \in X_{l+1}, v \in N'_{l+2}\). By Fact 2, \(y_{l+2}\) is not adjacent to \(x_{l+1}, x_{l+2}, x_{l+3}, u, v\). Hence \(deg(y_{l+2}) = 2\) which contradicts the fact that \(G\) is 3-edge-connected.

Subcase 2: \(u \in X_{l+1}, v \in N''_{l+2}\). By Fact 2 and since \(deg(u), deg(v) \geq 3\), \(u\) is adjacent to exactly \(x_l, x_{l+1}, x_{l+2}\) while \(v\) is adjacent to exactly \(y_{l+1}, y_{l+2}, y_{l+3}\), and there are no edges between \(\{u, x_l, x_{l+1}, x_{l+2}\}\) and \(\{v, y_{l+1}, y_{l+2}, y_{l+3}\}\). It follows that \(G\) can be disconnected by deleting the two edges \(y_{l+1}x_{l+1}, x_{l+2}x_{l+3}\), a contradiction. This completes the proof of Part B.

PART C: If \((N_l, N_{l+1})\) is a 5-class, then \((N_{l+3}, N_{l+4})\) is not a 5-class

Suppose to the contrary that both \((N_l, N_{l+1})\) and \((N_{l+3}, N_{l+4})\) are 5-classes. Let \(N_l \cup N_{l+1} = \{x_l, y_l, x_{l+1}, y_{l+1}, u\}\) and \(N_{l+3} \cup N_{l+4} = \{x_{l+3}, y_{l+3}, x_{l+4}, y_{l+4}, v\}\). By \(|E(N_l, N_{l+1})| \geq 3\) and since \(x_l y_{l+1}, y_{l+1} x_{l+1} \notin E(G)\), \(u\) is adjacent to \(\{y_l, y_{l+1}\}\) or \(\{x_l, x_{l+1}\}\) (but not both). Therefore, \(u \in Y_l \cup Y_{l+1}\) or \(u \in X_l \cup X_{l+1}\). Assume that \(u \in Y_l \cup Y_{l+1}\) (The case \(u \in X_l \cup X_{l+1}\) is proved analogously). A similar argument for \(v\) shows that \(v \in Y_{l+3} \cup Y_{l+4}\) or \(v \in X_{l+3} \cup X_{l+4}\). Now consider \(N_{l+2}\). Since each vertex in \(N_{l+2}\) is adjacent to \(N_{l+1} = X_{l+1} \cup Y_{l+1} = \{x_{l+1}\} \cup Y_{l+1}\), we have \(N_{l+2} = X_{l+2} \cup N''_{l+2}\). We have proved that \(\cup_{l+4}^l N_l = (\cup_{l+4}^l N'_{l}) \cup (\cup_{l+4}^l N''_l) = (\cup_{l+4}^l X_{l}) \cup (\cup_{l+4}^l Y_{l})\). By Fact 2, there are no edges between \((\cup_{l+4}^l N'_{l})\) and \((\cup_{l+4}^l N''_l)\). Therefore, the removal of \(x_l x_{l+1}, x_{l+3} x_{l+4}\) (if \(v \in Y_l \cup Y_{l+1}\)) or \(x_l x_{l+1}, y_{l+3} y_{l+4}\) (if \(v \in X_l \cup X_{l+1}\)) disconnects \(G\), a contradiction.

\[\square\]

Definition 4 A distance layer \(N_l\) is adjacent to a 5-class \((N_l, N_{l+1})\) if
\[ i \in \{l - 1, l, l + 1, l + 2\}. \]

**Lemma 4** Let \( \lambda(G) \geq 3, \ \text{rad}(G) \geq 10. \)

(a) Let \( i \in \{8, \cdots, r - 11\} \) be such that \( N_i \) is not adjacent to a 5-class. If \( k_i = 2, \) then \( k_{i-1} \geq 4 \) and \( k_{i-1} \geq 4. \)

(b) Let \( (N_i, N_{i+1}) \) be a 5-class and \( e \) be the least integer such that \( (N_{i+e}, N_{i+e+1}) \)

is a 5-class where \( l + e < r - 10. \) Then \[ \sum_{i=l+3}^{l+e-2} k_i \geq 3(e - 4). \]

**Proof:** (a) We already know that \( k_i + k_{i+1} \geq 5. \) Thus if \( k_i = 2, \) then \( k_{i+1} \geq 4 \) unless \( N_i \) is adjacent to a 5-class. Similarly \( k_{i-1} \geq 4. \)

(b) Assume the given condition. Since for each \( i \in \{l+3, l+4, \cdots, l+e-2\} \)

\((N_i, N_{i+1})\) is not a 5-class, from (a) above, we must have \( k_i + k_{i+1} \geq 6 \) for all \( i \in \{l+3, \cdots, l+e-3\}. \) Therefore, if \( e - 4 \) is even, then \[ \sum_{i=l+3}^{l+e-2} k_i = (k_{i+3} + k_{i+4}) + \cdots + (k_{l+e-3} + k_{l+e-2}) \geq \frac{e - 4}{2} \]
\[ \geq \frac{5}{2} \]
\[ \geq 3e - 13 \]

as desired. So assume that \( e - 4 \) is odd. Now, \[ \sum_{i=l+3}^{l+e-2} k_i = (k_{i+3} + k_{i+4}) + \cdots + (k_{l+e-3} + k_{l+e-2}) \geq \frac{e - 4}{2} \]
\[ \geq \frac{5}{2} \]
\[ \geq 3e - 13 \]

as desired. We show that the above inequalities are in fact strict. So suppose to the contrary that both inequalities hold with equality. Then \( k_{i+3} = k_{l+e-2} = 2 \) and \( k_i + k_{i+1} = 6 \) for all \( i \in \{l + 3, l + 4, \cdots, l + e - 3\}. \)

Hence \( (k_{i+3}, k_{i+4}, \cdots, k_{l+e-2}) = (2, 4, 2, 4, \cdots, 2, 4, 2, 4). \) Let \( N_i \cup N_{i+1} = \{x_i, y_i, x_{i+1}, y_{i+1}, u\} \) and \( N_{i+e} \cup N_{i+e+1} = \{x_{i+e}, y_{i+e}, x_{i+e+1}, y_{i+e+1}, v\}. \)

By 3-edge-connectedness and since \( y_ix_{i+1}, x_{i+1}y_{i+1} \notin E(G), u \) is adjacent to \( \{x_i, x_{i+1}\} \) or \( \{y_i, y_{i+1}\} \) (but not both). Hence \( u \in X_i \cup X_{i+1} \) or \( u \in Y_i \cup Y_{i+1}. \) Assume that \( u \in Y_i \cup Y_{i+1}, \) (The case \( u \in X_i \cup X_{i+1} \) is solved analogously). Since each vertex in \( N_{i+2} \) is adjacent to a vertex in \( N_{i+1} = X_{i+1} \cup Y_{i+1} = \{x_{i+1}, y_{i+1}\}, \) we have \( N_{i+2} = X_{i+2} \cup N'_{i+2}. \) Since for each \( i = l + 3, l + 4, l + 5, \cdots, l + e - 2, k_i = 2, \) then \( N_{i+1} = X_{i+1} \cup Y_{i+1}. \) Note also that by 3-edge-connectedness and since \( x_{i+e}y_{i+e+1}, y_{i+e}y_{i+e+1} \notin E(G), v \) is adjacent to \( \{y_{i+e}, y_{i+e+1}\} \) or \( \{x_{i+e}, x_{i+e+1}\} \) (but not both). Thus \( N_{i+e} \cup N_{i+e+1} = \{x_{i+e} \cup Y_{i+e} \} \cup \{x_{i+e+1} \cup Y_{i+e+1}\}. \) We have proved that \( \cup_{i=e}^{r-1} N_i = \cup_{i=e}^{r-1} N'_i = \cup_{i=e}^{r-1} X_i. \) Hence by Fact 2, there are no edges between \( \cup_{i=e}^{r-1} N_i \) and \( \cup_{i=e}^{r-1} N'_i = \cup_{i=e}^{r-1} X_i. \) Therefore, the removal of \( x_i x_{i+1}, x_{i+e}x_{i+e+1} \) (if \( v \in Y_i \cup Y_{i+1} \) or \( x_i x_{i+1}, y_{i+e}y_{i+e+1} \) (if \( v \in X_i \cup X_{i+1} \) disconnects \( G, \) a contradiction.

\[ \square \]

**Theorem 1** Let \( G \) be a 3-edge-connected graph of order \( n. \) Then \( \text{rad}(G) \leq \max\{9, \frac{1}{3} n + \frac{13}{2}\} \) and this bound is asymptotically sharp.

**Proof:** Let \( z \) be a centre vertex of \( G \) as before. Assume the notation for \( z_r \) and \( y_r. \) Let \( l_1, l_2 \in \{8, 9, \cdots, r - 11\} \) with \( l_1 \leq l_2 \) be the least and greatest
integers respectively such that $(N_{l_1}, N_{l_1+1})$ and $(N_{l_2}, N_{l_2+1})$ are 5-classes. Then by Lemma 3 and Lemma 4 (b), we have
\[
\sum_{i=1}^{l_2+2} k_i \geq 3(l_2 - l_1 + 4)
\] (2)

By Lemma 4 (a), $k_i + k_{i+1} \geq 6$ for all $i \in \{7, \ldots, l_1 - 2\} \cup \{l_2 + 3, \ldots, r - 9\}$. Thus
\[
\sum_{i=7}^{l_1-2} k_i = \begin{cases} 
(k_7 + k_8) + \cdots + (k_{l_1-3} + k_{l_1-2}) & \text{if } l_1 \text{ is even} \\
(k_7 + k_8) + \cdots + (k_{l_1-4} + k_{l_1-3}) + k_{l_1-2} & \text{if } l_1 \text{ is odd}
\end{cases}
\geq 3(l_1 - 9) + 2
\] (3)

Similarly,
\[
\sum_{i=l_2+3}^{r-9} k_i \geq 3(r - 12 - l_2) + 2.
\] (4)

Thus combining (2) (3) and (4) we obtain
\[
n \geq |N_{\leq 6}| + 3(l_1 - 9) + 2 + 3(l_2 - l_1 + 4) + 3(r - 12 - l_2) + 2 + |N_{\geq r-8}|
= |N_{\leq 6}| + 3r - 47 + |N_{\geq r-8}|.
\]

To find lower bounds to $|N_{\leq 6}|$ and $|N_{\geq r-8}|$, note that since $G$ is 3-edge-connected, for each $i = 0, 1, 2, \ldots, r - 1$, $k_i + k_{i+1} \geq 3$, hence $k_i + k_{i+1} \geq 4$. Therefore, $|N_{\leq 6}| \geq 13$ and $|N_{\geq r-8}| \geq 17$. It follows that $r \leq \frac{n}{3} + \frac{17}{3}$ as desired.

For positive integers $n, k$ with $n = \frac{3}{2}k$ and $k$ even, the 3-edge-connected graph $G = C_k(1, 2)$ has order $n$ and radius, $\lfloor \frac{k}{2} \rfloor = \lfloor \frac{1}{3}n \rfloor$. Therefore, the given bound is asymptotically sharp. \[\square\]

**References**


